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ELECTRICAL ENGINEERING RESEARCH LABORATORY
THE UNIVERSITY OF TEXAS
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Report No. 143

20 May 1966

A FINITE-DIFFERENCE STUDY OF THE EFFECT OF CURRENT AND CHARGE SOURCES ON THE ELECTROMAGNETIC FIELD OF THE EARTH

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Contract Nonr 375(14)
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ABSTRACT

A finite-difference approach to the description of the electromagnetic field structure within the region bounded by the earth and some outer spherical boundary is studied. Within this medium, a current sheet and a free charge sheet are assumed to exist in a spherical geometry.

The medium is assumed to be two-dimensionally inhomogeneous and to consist of a plasma gas. The earth is considered to be perfectly conducting, and the field distribution at the outer boundary is caused to assume the form of a system of spherical waves which are propagating outwardly.

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I. INTRODUCTION

An investigation into the nature of the electromagnetic field of the earth produced by both current and free charge sources located in the upper atmosphere is the basis of this problem.

The medium surrounding the earth is taken to be generally inhomogeneous and anisotropic. The two spherical boundaries which enclose the cavity of interest are the earth or the earth's core for the inner boundary and the magnetospheric boundary for the outer confinement of the fields. It may, however, be desirable for a different boundary to be chosen for the outer limits of the problem.

At some altitude is postulated a current sheet and a sheet of free surface charge. Both will initially be assumed to be positioned at the same altitude in order that the notation used may be simplified. The fields within the boundaries are mathematically examined.

II. MATHEMATICAL DEVELOPMENT

Mathematical analysis is now performed on the problem. Because of the geometry of the system, spherical coordinates are used.

A. The Working Differential Equations

The appropriate place to begin the problem is the group of Maxwell's equations which, written in the standard vector notation, are

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \cdot \vec{D} = \rho . \tag{4}$$

For the purposes of this problem, it is assumed that the displacement current, $\frac{\partial \vec{D}}{\partial t}$, is small enough compared to the free current, $\vec{J}$, so as to be negligible. This approximation is generally valid for the media under consideration if the frequencies used are small. However, the approximation is only valid within plasma regions and does not hold with the relatively thin air layer. The frequencies of interest are those frequencies of the micropulsations in the range from about 0.001 cycles per second to a maximum of 1 cycle per second. This assumption reduces equation (1) to

$$\nabla \times \vec{H} = \vec{J} . \tag{5}$$

For a medium which is linear (non-magnetic), the magnetic flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ by the permeability constant of the material,

$$B = \mu \vec{H} .$$

If the permeability is not a function of time, then equation (2) becomes

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} . \tag{6}$$

The time dependence of all field quantities is assumed to be of the form $e^{i\omega t}$. Thus,

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{i\omega t}$$

$$\vec{H}(\vec{r}, t) = \vec{H}(\vec{r}) e^{i\omega t} .$$

There is no loss of generality in this assumption since any other forms of time variations may be constructed from the assumed mode by a Fourier series.

The curl equation (6) is now of the form

$$\nabla \times \vec{E} = - i\omega\mu\vec{H} . \tag{7}$$

The factor $e^{i\omega t}$ is understood to be associated with each of the field quantities and is henceforth not written.

For media which are anisotropic, the conductivity is usually expressed as a dyadic. The free current density is then given by the vector operation

$$\vec{J} = \overline{\overline{\sigma}} \cdot \vec{E} .$$

In spherical coordinates this operation is expanded into the three components as

$$J_r = \sigma_{rr}E_r + \sigma_{r\theta}E_\theta + \sigma_{r\phi}E_\phi$$

$$J_\theta = \sigma_{\theta r}E_r + \sigma_{\theta\theta}E_\theta + \sigma_{\theta\phi}E_\phi$$

$$J_\phi = \sigma_{\phi r}E_r + \sigma_{\phi\theta}E_\theta + \sigma_{\phi\phi}E_\phi .$$

If the curl operations are performed in this coordinate system, equations (5) and (7) become, in component form,

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right] = \sigma_{rr}E_r + \sigma_{r\theta}E_\theta + \sigma_{r\phi}E_\phi \tag{8}$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial}{\partial r}(r H_\phi)\right] = \sigma_{\theta r}E_r + \sigma_{\theta\theta}E_\theta + \sigma_{\theta\phi}E_\phi \tag{9}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial\theta}\right] = \sigma_{\phi r}E_r + \sigma_{\phi\theta}E_\theta + \sigma_{\phi\phi}E_\phi \tag{10}$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(E_\phi \sin\theta) - \frac{\partial E_\theta}{\partial\phi}\right] = -i\omega\mu H_r \tag{11}$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}(r E_\phi)\right] = -i\omega\mu H_\theta \tag{12}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial\theta}\right] = -i\omega\mu H_\phi \tag{13}$$

These are the general equations which are made use of in the solution of this problem.

As they stand, equations (8) through (13) are rather unmanageable in the three dimensions. If it is assumed that the conductivity

is not functionally related to the azimuth angle ϕ, a simplification may be made which allows a type of separation of the preceding equations. For this case it is assumed that a symmetry exists which allows the fields to be separated into the form of

$$E(r, \theta, \phi) = E(r, \theta) e^{im\phi}$$

$$H(r, \theta, \phi) = H(r, \theta) e^{im\phi} .$$

When applied to equations (8) through (13), this assumption yields, with the factor $e^{im\phi}$ understood,

$$\frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - imH_\theta\right] = \sigma_{rr}E_r + \sigma_{r\theta}E_\theta + \sigma_{r\phi}E_\phi \tag{14}$$

$$\frac{1}{r}\left[\frac{im}{\sin\theta} H_r - \frac{\partial}{\partial r}(r H_\phi)\right] = \sigma_{\theta r}E_r + \sigma_{\theta\theta}E_\theta + \sigma_{\theta\phi}E_\phi \tag{15}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r H_\theta) - \frac{\partial H_r}{\partial\theta}\right] = \sigma_{\phi r}E_r + \sigma_{\phi\theta}E_\theta + \sigma_{\phi\phi}E_\phi \tag{16}$$

$$\frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(E_\phi \sin\theta) - imE_\theta\right] = -i\omega\mu H_r \tag{17}$$

$$\frac{1}{r}\left[\frac{im}{\sin\theta} E_r - \frac{\partial}{\partial r}(r E_\phi)\right] = -i\omega\mu H_\theta \tag{18}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial\theta}\right] = -i\omega\mu H_\phi . \tag{19}$$

The derivatives in equations (14) through (19) are now expanded, and the terms are rearranged to give

$$H_r = -\frac{\cos\theta}{i\omega\mu r \sin\theta} E_\phi - \frac{1}{i\omega\mu r}\frac{\partial E_\phi}{\partial\theta} + \frac{im}{i\omega\mu r \sin\theta} E_\theta \tag{20}$$

$$E_r = \frac{1}{\sigma_{rr}}\left[\frac{\cos\theta}{r\sin\theta}H_\phi + \frac{1}{r}\frac{\partial H_\phi}{\partial\theta} - \frac{im}{r\sin\theta}H_\theta - \sigma_{r\theta}E_\theta - \sigma_{r\phi}E_\phi\right] \tag{21}$$

$$\frac{\partial H_\theta}{\partial r} = \sigma_{\phi r}E_r + \sigma_{\phi\theta}E_\theta + \sigma_{\phi\phi}E_\phi + \frac{1}{r}\frac{\partial H_r}{\partial\theta} - \frac{1}{r}H_\theta \tag{22}$$

$$\frac{\partial E_\theta}{\partial r} = \frac{1}{r}\frac{\partial E_r}{\partial\theta} - \frac{1}{r}E_\theta - i\omega\mu H_\phi \tag{23}$$

$$\frac{\partial H_\phi}{\partial r} = -\sigma_{\theta r}E_r - \sigma_{\theta\theta}E_\theta - \sigma_{\theta\phi}E_\phi - \frac{im}{r\sin\theta}H_r - \frac{1}{r}H_\phi \tag{24}$$

$$\frac{\partial E_\phi}{\partial r} = i\omega\mu H_\theta + \frac{im}{r\sin\theta}H_r - \frac{1}{r}E_\phi . \tag{25}$$

The radial fields are therefore expressed in terms of the tangential fields and their derivatives, while the derivatives of the tangential fields in the radial direction occur in terms of the fields and their tangential derivatives only. A method for the calculation of the fields now presents itself.

Consider a quadrant of the constant ϕ-plane, as shown in Figure 1, partitioned by a grid of N points in the θ-direction by M points in the radial direction. Between level L (levels referring to the radial direction) and level L+1, a boundary exists which carries a surface current $\vec{J}_s$ and a free surface charge distribution ρ_s.

If the tangential fields on the inner boundary are completely specified, then their derivatives with respect to the polar angle θ are

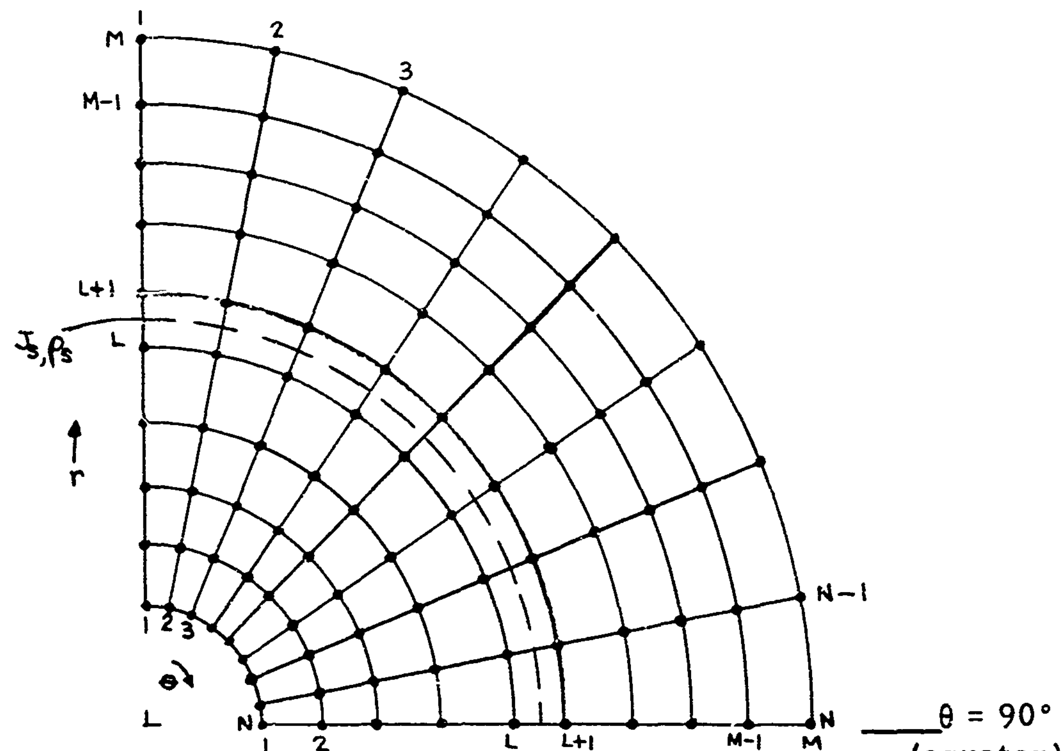

Figure 1

Quadrant of Constant ϕ-Plane ρ - r grid

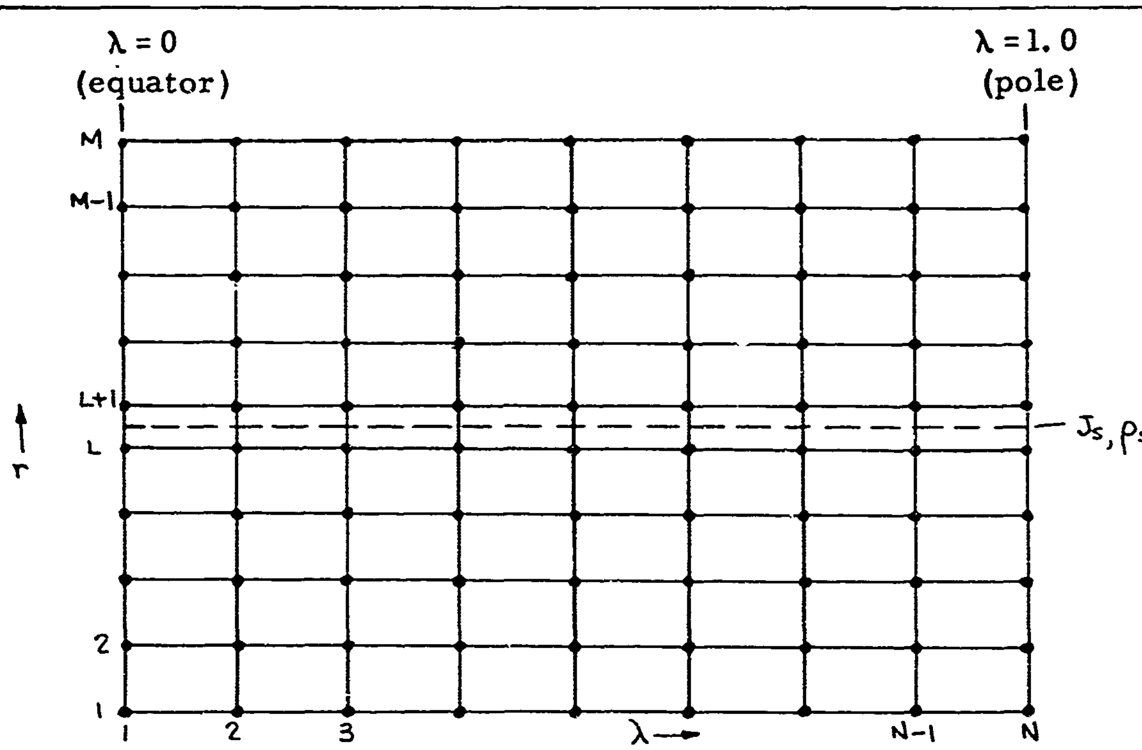

Figure 2

Quadrant of Constant ϕ-Plane λ - r grid

specified, and equations (20) and (21) may be utilized for the determination of the two radial field components at all points on the inner boundary. Integration of the equations (22) through (25) then yields the tangential fields at the second level. This process is continued until the fields at level L are known. After proper boundary conditions are applied at the discontinuity between levels L and L+1, the fields at level L+1 are determined, and the marching process is continued through level M.

The method may, of course, be employed for the entire range of polar angle θ from 0° to 180°. However, the summetry of the earth's dipole field about the equatorial plane leads to a simplification in that fewer points are needed if the plane of $\theta = 90°$ is thought of as an artificial boundary. The field quantities are then terminated at this boundary as odd functions about $\theta = 90°$ (in which case the field is zero at the boundary) or even functions about the equator (in which case the derivatives of the fields with respect to the polar angle is zero).

If the radial component of the electric field is chosen as being an even function of θ about $\theta = 90°$, then it can be shown that H_θ and E_ϕ must also be even functions about the equator while H_r, E_θ and H_ϕ are odd functions. The converse is also true. It is also true that the integer m specifies a symmetry of the fields about the pole.

Some difficulty may arise in the use of the differential equations (20) through (25) at points near the equatorial axis if equal

increments of the angle theta are used. In order that more points be located near this axis, giving greater resolution in this area of trouble, the independent variable theta is changed to cos θ. The two-dimensional space under consideration may now be represented as in Figure 2.

The transformation from angle increments to increments based on the cosine of the angle is effected by the substitution of $\lambda = \cos\theta$ into equations (20) through (25). Then,

$$\sin\theta = \sqrt{1-\lambda^2},$$

and

$$\frac{\partial}{\partial\theta} = -\sqrt{1-\lambda^2}\,\frac{\partial}{\partial\lambda}.$$

The working equations now become

$$H_r = -\frac{\lambda}{i\omega\mu r\sqrt{1-\lambda^2}}E_\phi + \frac{\sqrt{1-\lambda^2}}{i\omega\mu r}\frac{\partial E_\phi}{\partial\lambda} + \frac{m}{\omega\mu r\sqrt{1-\lambda^2}}E_\theta \tag{26}$$

$$E_r = \frac{1}{\sigma_{rr}}\left[\frac{\lambda}{r\sqrt{1-\lambda^2}}H_\phi - \frac{\sqrt{1-\lambda^2}}{r}\frac{\partial H_\phi}{\partial\lambda} - \frac{im}{r\sqrt{1-\lambda^2}}H_\theta - \sigma_{r\theta}E_\theta - \sigma_{r\phi}E_\phi\right] \tag{27}$$

$$\frac{\partial H_\theta}{\partial r} = \sigma_{\phi r}E_r + \sigma_{\phi\theta}E_\theta + \sigma_{\phi\phi}E_\phi - \frac{\sqrt{1-\lambda^2}}{r}\frac{\partial H_r}{\partial\lambda} - \frac{1}{r}H_\theta \tag{28}$$

$$\frac{\partial E_\theta}{\partial r} = -\frac{\sqrt{1-\lambda^2}}{r}\frac{\partial E_r}{\partial\lambda} - \frac{1}{r}E_\theta - i\omega\mu H_\phi \tag{29}$$

$$\frac{\partial H_\phi}{\partial r} = -\sigma_{\theta r}E_r - \sigma_{\theta\theta}E_\theta - \sigma_{\theta\phi}E_\phi - \frac{im}{r\sqrt{1-\lambda^2}}H_r - \frac{1}{r}H_\phi \tag{30}$$

$$\frac{\partial E_\phi}{\partial r} = i\omega\mu H_\theta + \frac{im}{r\sqrt{1-\lambda^2}} E_r - \frac{1}{r} E_\phi . \tag{31}$$

Now equal increments of λ are used instead of the θ increments.

B. Finite Difference Equations

The most straightforward manner for the solution of differential equations such as equations (26) through (31) on a high-speed digital computer is the finite difference approach. The finite-differencing of equations (26) through (31) follows with the aid of standard formulae found in Appendix A. The equations are then written as

$$H_r(p,q) = -\frac{\lambda}{i\omega\mu r\sqrt{1-\lambda^2}} E_\phi(p,q) + \frac{\sqrt{1-\lambda^2}}{i\omega\mu r}\left[\frac{E_\phi(p,q+1) - E_\phi(p,q-1)}{2\Delta\lambda}\right] + \frac{m}{\omega\mu r\sqrt{1-\lambda^2}} E_\theta(p,q) \tag{32}$$

$$E_r(p,q) = \frac{1}{\sigma_{rr}}\left\{\frac{\lambda}{r\sqrt{1-\lambda^2}} H_\phi(p,q) - \frac{\sqrt{1-\lambda^2}}{r}\left[\frac{H_\phi(p,q+1) - H_\phi(p,q-1)}{2\Delta\lambda}\right] - \frac{im}{r\sqrt{1-\lambda^2}} H_\theta(p,q) - \sigma_{r\theta}E_\theta(p,q) - \sigma_{r\phi}E_\phi(p,q)\right\} \tag{33}$$

$$\frac{H_\theta(p+1,q) - H_\theta(p-1,q)}{2\Delta r} = \sigma_{\phi r}E_r(p,q) + \sigma_{\phi\theta}E_\theta(p,q) + \sigma_{\phi\phi}E_\phi(p,q) - \frac{\sqrt{1-\lambda^2}}{r}\left[\frac{H_r(p,q+1) - H_r(p,q-1)}{2\Delta\lambda}\right] - \frac{1}{r}H_\theta(p,q) \tag{34}$$

$$\frac{E_\theta(p+1,q) - E_\theta(p-1,q)}{2\Delta r} = -\frac{\sqrt{1-\lambda^2}}{r}\,\frac{E_r(p,q+1) - E_r(p,q-1)}{2\Delta\lambda} - \frac{1}{r}E_\theta(p,q) - i\omega\mu H_\phi(p,q) \tag{35}$$

$$\frac{H_\phi(p+1,q) - H_\phi(p-1,q)}{2\Delta r} = -\sigma_{\theta r}E_r(p,q) - \sigma_{\theta\theta}E_\theta(p,q) - \sigma_{\theta\phi}E_\phi(p,q) - \frac{im}{r\sqrt{1-\lambda^2}}H_r(p,q) - \frac{1}{r}H_\phi(p,q) \tag{36}$$

$$\frac{E_\phi(p+1,q) - E_\phi(p-1,q)}{2\Delta r} = i\omega\mu\; H_\theta(p,q) + \frac{im}{r\sqrt{1-\lambda^2}}E_r(p,q) - \frac{1}{r}E_\phi(p,q)\,. \tag{37}$$

In equations (32) through (37) all conductivities are those at the point (p, q), the λ is λ_q, and r is r_p.

The final form of the difference equations follows from the solution of equations (34) through (37) for the tangential fields at the layer p+1. The results are the following equations:

$$H_r(p,q) = -\frac{\lambda}{i\omega\mu r\sqrt{1-\lambda^2}}E_\phi(p,q) + \frac{\sqrt{1-\lambda^2}}{2i\omega\mu r\,\Delta\lambda}E_\phi(p,q+1) - \frac{\sqrt{1-\lambda^2}}{2i\omega\mu r\,\Delta\lambda}E_\phi(p,q-1) + \frac{m}{\omega\mu r\sqrt{1-\lambda^2}}E_\theta(p,q) \tag{38}$$

$$E_r(p,q) = \frac{\lambda}{\sigma_{rr}\, r\sqrt{1-\lambda^2}} H_\phi(p,q) - \frac{\sqrt{1-\lambda^2}}{2\sigma_{rr}\, r\,\Delta\lambda} H_\theta(p,q+1)$$

$$- \frac{\sqrt{1-\lambda^2}}{2\sigma_{rr}\, r\,\Delta\lambda} H_\phi(p,q-1) + \frac{m}{\omega\mu r\sqrt{1-\lambda^2}} E_\theta(p,q) \qquad (39)$$

$$H_\theta(p+1,q) = H_\theta(p-1,q) + 2\Delta r\, \sigma_{\phi r} E_r(p,q) + 2\Delta r\, \sigma_{\phi\theta} E_\theta(p,q)$$

$$+ 2\Delta r\, \sigma_{\phi\phi} E_\phi(p,q) - \frac{\Delta r\sqrt{1-\lambda^2}}{r\Delta\lambda} H_r(p,q+1)$$

$$+ \frac{\Delta r\sqrt{1-\lambda^2}}{r\Delta\lambda} H_r(p,q-1) - \frac{2\Delta r}{r} H_\theta(p,q) \qquad (40)$$

$$E_\theta(p+1,q) = E_\theta(p-1,q) - \frac{\Delta r\sqrt{1-\lambda^2}}{r\Delta\lambda} E_r(p,q+1) + \frac{\Delta r\sqrt{1-\lambda^2}}{r\Delta\lambda} E_r(p,q-1)$$

$$- \frac{2\Delta r}{r} E_\theta(p,q) - 2i\omega\mu\,\Delta r\, H_\phi(p,q) \qquad (41)$$

$$H_\phi(p+1,q) = H_\phi(p-1,q) - 2\sigma_{\theta r}\,\Delta r\, E_r(p,q) - 2\Delta r\, \sigma_{\theta\theta} E_\theta(p,q)$$

$$- 2\Delta r\, \sigma_{\theta\phi} E_\phi(p,q) - \frac{2\,im\,\Delta r}{r\sqrt{1-\lambda^2}} H_r(p,q) \qquad (42)$$

$$- \frac{2\Delta r}{r} H_\phi(p,q)$$

$$E_\phi(p+1,q) = E_\phi(p-1,q) + 2i\omega\mu\Delta r\, H_\theta(p,q) + \frac{2im\Delta r}{r\sqrt{1-\lambda^2}} E_r(p,q) - \frac{2\Delta r}{r} E_\phi(p,q).$$

$$(43)$$

The functional form of the difference equations is

$$H_r(p,q) = f\left[E_\phi(p,q),\ E_\phi(p,q+1),\ E_\phi(p,q-1),\ E_\theta(p,q)\right] \tag{44}$$

$$E_r(p,q) = f\left[H_\phi(p,q),\ H_\phi(p,q+1),\ H_\phi(p,q-1),\ H_\theta(p,q),\ E_\theta(p,q),\ E_\phi(p,q)\right] \tag{45}$$

$$H_\theta(p+1,q) = f\left[H_\theta(p-1,q),\ E_r(p,q),\ E_\theta(p,q),\ E_\phi(p,q)\ H_r(p,q+1),\ H_r(p,q-1),\ H_\theta(p,q)\right] \tag{46}$$

$$E_\theta(p+1,q) = f\left[E_\theta(p-1,q),\ E_r(p,q+1),\ E_r(p,q-1),\ E_\theta(p,q),\ H_\phi(p,q)\right] \tag{47}$$

$$H_\phi(p+1,q) = f\left[H_\phi(p-1,q),\ E_r(p,q),\ E_\theta(p,q),\ E_\phi(p,q)\ H_r(p,q),\ H_\phi(p,q)\right] \tag{48}$$

$$E_\phi(p+1,q) = f\left[E_\phi(p-1,q),\ H_\theta(p,q),\ E_r(p,q),\ E_\phi(p,q)\right]. \tag{49}$$

Since the forward difference is used in the initial step, $H_\theta(p-1,q)$, $E_\theta(p-1,q)$, $H_\phi(p-1,q)$ and $E_\phi(p-1,q)$ in equations (44) through (49) are replaced with $H_\theta(p,q)$, $E_\theta(p,q)$, $E_\theta(p,q)$, $H_\phi(p,q)$ and $E_\phi(p,q)$, respectively, although the exact equations change by a factor of two. Because of the symmetry conditions imposed, the central difference formula will also be used at $\lambda = 0$ and $\lambda = 1$.

Initially all tangential fields at the inner boundary are set equal to zero except $H_\theta(1, 1)$. From equation (38) it is seen that $H_r(1, q)$ is zero for all q. Equation (39) is used to determine $E_r(1, q)$ as a function of $H_\theta(1, 1)$, all other tangential fields on that level being zero. Equations (40) through (43) are then applied to yield the tangential fields at all points on the second radial level in terms of $H_\theta(1, 1)$, since all quantities on the right-hand side of these four equations are either zero or are in terms of $H_\theta(1, 1)$.

With $p = 2$ for the second level, equations (44) and (45) yield the radial fields $E_r(2, q)$ and $H_r(2, q)$ as a linear function of $H_\theta(1, 1)$ and, therefore, all fields at the second level are specified as a function of $H_\theta(1, 1)$. The process of obtaining the six field components at each level is continued until all fields directly below the current sheet and charge distribution are known in terms of $H_\theta(1, 1)$.

In order that as much accuracy as possible be retained in the calculations, the level below and the level above the discontinuity should be relatively close to this level as in Figures 1 and 2.

A general discontinuity is represented in Figure 3. Surface current and surface charge are present on the boundary.

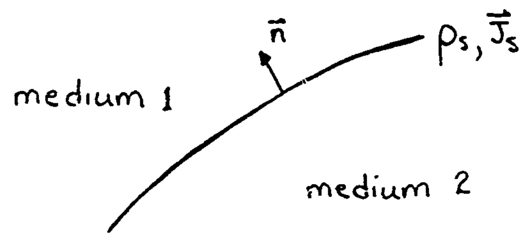

Figure 3.
A General Discontinuity

The boundary conditions for the fields around such a discontinuity are

$$\vec{n} \cdot (\vec{H}_1 - \vec{H}_2) = 0 \tag{50}$$

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = \rho_s/\epsilon_o \tag{51}$$

$$\vec{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \tag{52}$$

$$\vec{n} \times (\vec{E}_1 - \vec{E}_2) = 0 . \tag{53}$$

The unit vector $\vec{n}$ for this problem is the radial unit vector $\vec{a}_r$. Expansion of these four equations in spherical coordinates gives

$$H_{r1} - H_{r2} = 0 \qquad\qquad E_{r1} - E_{r2} = \rho_s/\epsilon_o$$

$$H_{\theta 1} - H_{\theta 2} = J_{s\phi} \qquad\qquad E_{\theta 1} - E_{\theta 2} = 0$$

$$H_{\phi 1} - H_{\phi 2} = -J_{s\theta} \qquad\qquad E_{\phi 1} - E_{\phi 2} = 0 ,$$

or, in terms applicable to Figure 2,

$$H_r(L+1, q) = H_r(L, q)$$

$$H_\theta(L+1, q) = H_\theta(L, q) + J_{s\phi}(q)$$

$$H_\phi(L+1, q) = H_\phi(L, q) - J_{s\theta}(q)$$

$$E_r(L+1, q) = E_r(L, q) + \rho_s(q)/\varepsilon_o$$

$$E_\theta(L+1, q) = E_\theta(L, q)$$

$$E_\phi(L+1, q) = E_\phi(L, q) \quad ,$$

for all q, $1 \leq q \leq N$.

These expressions relate the six field components at level L+1 to the six components at level L. Since those at level L are already known in terms of $H_\theta(1, 1)$, both tangential and radial fields are now related to $H_\theta(1, 1)$ and the source terms directly below each point on the discontinuity.

Symbolically,

$$E_r(L+1, q) = f\left[H_\theta(1, 1)\ ,\ \rho_s(q)\right]$$

$$E_\theta(L+1, q) = f\left[H_\theta(1, 1)\right]$$

$$E_\phi(L+1, q) = f\left[H_\theta(1, 1)\right]$$

$$H_r(L+1, q) = f\left[H_\theta(1, 1)\right]$$

$$H_\theta(L+1, q) = f\left[H_\theta(1, 1)\ ,\ J_{s\phi}(q)\right]$$

$$H_\phi(L+1, q) = f\left[H_\theta(1, 1),\ J_{s\theta}(q)\right] .$$

The functions are linear with respect to $H_\theta(1, 1)$, $\rho_s(q)$, $J_{s\phi}(q)$, and $J_{s\theta}(q)$. Now, from equation (46) through (49), the tangential fields at level L+2 are, in functional form,

$$H_\theta(L+2, q) = f\Big[H_\theta(L, q), \; E_r(L+1, q), \; E_\theta(L+1, q),$$

$$E_\phi(L+1, q), \; H_r(L+1, q+1), \; H_r(L+1, q-1),$$

$$H_\theta(L+1, q)\Big] \tag{54}$$

$$E_\theta(L+2, q) = f\Big[E_\theta(L, q), \; E_r(L+1, q+1), \; E_r(L+1, q-1),$$

$$E_\theta(L+1, q), \; H_\phi(L+1, q)\Big] \tag{55}$$

$$H_\phi(L+2, q) = f\Big[H_\phi(L, q), \; E_r(L+1, q), \; E_\theta(L+1, q),$$

$$E_\phi(L+1, q), \; H_r(L+1, q), \; H_\phi(L+1, q)\Big] \tag{56}$$

$$E_\phi(L+2, q) = f\Big[E_\phi(L, q), \; H_\theta(L+1, q), \; E_r(L+1, q),$$

$$E_\phi(L+1, q)\Big]. \tag{57}$$

Since

$$E_r(L+1, q+1) = f\Big[H_\theta(1, 1), \; \rho_s(q+1)\Big]$$

$$E_r(L+1, q-1) = f\Big[H_\theta(1, 1), \; \rho_s(q-1)\Big],$$

$$H_\theta(L+2, q) = f\Big[H_\theta(1, 1), \; J_{s\phi}(q), \; \rho_s(q)\Big]$$

$$E_\theta(L+2, q) = f\Big[H_\theta(1, 1), \; \rho_s(q+1), \; \rho_s(q-1), \; J_{s\theta}(q)\Big]$$

$$H_\phi(L+2, q) = f\Big[H_\theta(1, 1), \; J_{s\theta}(q), \; \rho_s(q)\Big]$$

$$E_\phi(L+2, q) = f\Big[H_\theta(1, 1), \; J_{s\phi}(q), \; \rho_s(q)\Big],$$

so that by equations (44) and (45),

$$H_r(L+2,q) = f\Big[H_\theta(1,1),\ J_{s\phi}(q),\ \rho_s(q),\ J_{s\phi}(q+1),$$
$$\rho_s(q+1),\ J_{s\phi}(q-1),\ \rho_s(q-1),\ J_{s\theta}(q)\Big]$$

$$E_r(L+2,q) = f\Big[H_\theta(1,1),\ J_{s\theta}(q),\ \rho_s(q),\ J_{s\theta}(q+1),$$
$$\rho_s(q+1),\ J_{s\theta}(q-1),\ \rho_s(q-1),\ J_{s\phi}(q)\Big].$$

It can be seen that, as more and more steps are taken away from the sources, the fields at angular location of λ become functions of more and more of the source components, so that, at level M (outer boundary) the fields may be functions of all the source terms. Thus

$$H_\theta(M,q) = f\Big[H_\theta(1,1),\ J_{s\theta}(1),\ J_{s\theta}(2),\ \cdots,\ J_{s\theta}(N),\ J_{s\phi}(1),$$
$$J_{s\phi}(2),\ \cdots,\ J_{s\phi}(N),\ \rho_s(1),\ \cdots,\ \rho_s(N)\Big],$$

and likewise for the remaining three tangential components. If all tangential fields at the inner boundary are zero except $H_\theta(1,2)$, then similar results are obtained as

$$H_\theta(M,q) = f\Big[H_\theta(1,2),\ J_{s\phi}(n),\ J_{s\theta}(n),\ \rho_s(n)\Big],$$

$n = 1,\ 2,\ \cdots,\ N.$

In this manner the outer tangential fields are found as functions of each of the tangential magnetic fields at the inner boundary and the source terms at each point of the discontinuity. The results may be noted as

$$
\begin{aligned}
E_\theta(M, q) = {} & A_{q1} H_\theta(1, 1) + \cdots + A_{qN} H_\theta(1, N) + A'_{q1} H_\phi(1, 1) + \cdots + \\
& A'_{qN} H_\phi(1, N) + \alpha_{q1} J_{s\phi}(1) + \cdots + \alpha_{qN} J_{s\phi}(N) + \alpha'_{q1} J_{s\theta}(1) \\
& + \cdots + \alpha'_{qN} J_{s\theta}(N) + \beta_{q1} \rho_s(1) + \cdots + \beta_{qN} \rho_s(N)
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
E_\phi(M, q) = {} & B_{q1} H_\theta(1, 1) + \cdots + B_{qN} H_\theta(1, N) + B'_{q1} H_\phi(1, 1) + \cdots + \\
& B'_{qN} H_\phi(1, N) + \gamma_{q1} J_{s\phi}(1) + \cdots + \gamma_{qN} J_{s\phi}(N) + \gamma'_{q1} J_{s\theta}(1) \\
& + \cdots + \gamma'_{qN} J_{s\theta}(N) + \delta_{q1} \rho_s(1) + \cdots + \delta_{qN} \rho_s(N)
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
H_\theta(M, q) = {} & C_{q1} H_\theta(1, 1) + \cdots + C_{qN} H_\theta(1, N) + C'_{q1} H_\phi(1, 1) + \cdots + \\
& C'_{qN} H_\phi(1, N) + \xi_{q1} J_{s\phi}(1) + \cdots + \xi_{q1} J_{s\phi}(N) + \xi'_{q1} J_{s\theta}(1) \\
& + \cdots + \xi'_{qN} J_{s\theta}(N) + \eta_{q1} \rho_s(1) + \cdots + \eta_{qN} \rho_s(N)
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
H_\phi(M, q) = {} & D_{q1} H_\theta(1, 1) + \cdots + D_{qN} H_\theta(1, N) + D'_{q1} H_\phi(1, 1) + \cdots + \\
& D'_{qN} H_\phi(1, N) + \zeta_{q1} J_{s\phi}(1) + \cdots + \zeta_{qN} J_{s\phi}(N) + \zeta'_{q1} J_{s\theta}(1) \\
& + \cdots + \zeta'_{qN} J_{s\theta}(N) + \nu_{q1} \rho_s(1) + \cdots + \nu_{qN} \rho_s(N).
\end{aligned} \tag{61}
$$

In order to conveniently determine the coefficients in equations (58) through (61), we shall have to treat the source terms $J_{s\phi}$, $J_{s\theta}$

and ρ_s as unknowns. Then each coefficient is determined by setting all variables to zero except the variable associated with the coefficient desired, which is given the value of unity.

For instance, to determine the value of the 4N coefficients A_{q1}, B_{q1}, C_{q1}, D_{q1}; $q = 1, 2, \ldots, N$; all fields H_θ and H_ϕ at the inner boundary points are set to zero except $H_\theta(1, 1)$ which is unity, and all sources are assumed zero. Then

$$A_{q1} = E_\theta(M, q)$$

$$B_{q1} = E_\phi(M, q)$$

$$C_{q1} = H_\theta(M, q)$$

$$D_{q1} = H_\phi(M, q)$$

so that the coefficients are determined from the outer boundary fields produced by only a unity $H_\theta(1, 1)$. Similarly, α_{q1}, γ_{q1}, ξ_{q1} and ζ_{q1} are found by letting all tangential magnetic fields at the inner boundary be zero, and all source terms be zero except $J_{s\phi}(1)$.

In this manner all the coefficients are determined, and the outer tangential fields are found as known functions of the tangential magnetic fields at the inner boundary and the source distribution.

Equations (58) through (61) may be rewritten in more compact form as

$$E_\theta(M, q) = \sum_{n=1}^{N} \Big[A_{qn} H_\theta(1, n) + A'_{qn} H_\phi(1, n)$$

$$+ \alpha_{qn} J_{s\phi}(n) + \alpha'_{qn} J_{s\theta}(n) + \beta_{qn} \rho_s(n) \Big] \tag{62}$$

$$E_\phi(M, q) = \sum_{n=1}^{N} \Big[B_{qn} H_\theta(1, n) + B'_{qn} H \;\; (1, n)$$

$$+ \gamma_{qn} J_{s\phi}(n) + \gamma'_{qn} J_{s\theta}(n) + \delta_{qn} \rho_s(n) \Big] \tag{63}$$

$$H_\theta(M, q) = \sum_{n=1}^{N} \Big[C_{qn} H_\theta(1, n) + C'_{qn} H_\phi(1, n)$$

$$+ \xi_{qn} J_{s\phi}(n) + \xi'_{qn} J_{s\theta}(n) + \eta_{qn} \rho_s(n) \Big] \tag{64}$$

$$H_\phi(M, q) = \sum_{n=1}^{N} \Big[D_{qn} H_\theta(1, n) + D'_{qn} H_\phi(1, n)$$

$$+ \zeta_{qn} J_{s\phi}(n) + \zeta'_{qn} J_{s\theta}(n) + \nu_{qn} \rho_s(n) \Big] \tag{65}$$

Given a means for determining the tangential fields at one of the two boundaries and the source distribution, one is able to calculate the tangential fields at the other boundary and therefore all the fields between the earth and magnetosphere or within the constant dipole field cavity surrounding the earth. Equations (62) through (65) may also be written in matrix form,

$$\begin{bmatrix} E_\theta(M, q) \\ E_\phi(M, q) \\ H_\theta(Mlq) \\ H_\phi(M, q) \end{bmatrix} = \begin{bmatrix} A_{q1} \cdots A_{qN} & A'_{q1} \cdots A'_{qN} \\ B_{q1} \cdots B_{qN} & B'_{q1} \cdots B'_{qN} \\ C_{q1} \cdots C_{qN} & C'_{q1} \cdots C'_{qN} \\ D_{q1} \cdots D_{qN} & D'_{q1} \cdots D'_{qN} \end{bmatrix} \begin{bmatrix} H_\theta(1, 1) \\ \vdots \\ H_\theta(1, N) \\ H_\phi(1, 1) \\ \vdots \\ H_\phi(1, N) \end{bmatrix} + \Delta_1 \tag{66}$$

where

$$\Delta_1 = \begin{bmatrix} \alpha_{q1} \cdots \alpha_{qN} & \alpha'_{q1} \cdots \alpha'_{qN} & \beta_{q1} \cdots \beta_{qN} \\ \gamma_{q1} \cdots \gamma_{qN} & \gamma'_{q1} \cdots \gamma'_{qN} & \delta_{q1} \cdots \delta_{qN} \\ \xi_{q1} \cdots \xi_{qN} & \xi'_{q1} \cdots \xi'_{qN} & \eta_{q1} \cdots \eta_{qN} \\ \zeta_{q1} \cdots \zeta_{qN} & \zeta'_{q1} \cdots \zeta'_{qN} & \nu_{q1} \cdots \nu_{qN} \end{bmatrix} \begin{bmatrix} J_{s\phi}(1) \\ \vdots \\ J_{s\phi}(N) \\ J_{s\theta}(1) \\ \vdots \\ J_{s\theta}(N) \\ \rho_s(1) \\ \vdots \\ \rho_s(N) \end{bmatrix}$$

and q = 1, 2, . . . , N.

III. THE FORM OF THE FIELDS AT THE OUTER BOUNDARY

In some manner the tangential fields at the outer boundary must be determined in order that the matrix (66) may be inverted for the solution of the magnetic field intensities at all points on the surface of the inner spherical boundary. The finite-difference equations (38) through (43) may then be utilized for the final resolution of all fields between the two radial boundaries.

Two alternatives are available for the specification of the outer boundary fields. If the scope of the investigation includes the effect of the boundary conditions above on the field configuration within the confines of the radial limits, various distributions may be assigned to the tangential fields at the outer boundary. The problem then develops as to what physical significance the chosen boundary condition has. This procedure is particularly applicable to the case in which the exterior boundary is considered to be a conductor supporting a surface current.

The alternative to this specification is the mathematical generation of functions which might conceivably describe the form of the fields at the outer boundary. In order that this may be achieved, an investigation of the electromagnetic waves produced in a spherical system is made.

The field configuration at the outer limits of the region of interest is assumed to be of the form of spherical waves propagating radilly outward. The wave is considered to have propagated a sufficient distance that the medium in which the fields are examined may be assumed to be similar to free space with very small free electron densities.

A. Solutions to the Vector Wave Equation

If the displacement currents are retained in the Maxwell equations, then

$$\nabla \times \vec{H} = (\sigma + i\omega\epsilon)\vec{E}. \tag{67}$$

The curl of equation (7) is

$$\nabla \times \nabla \times \vec{E} = -i\omega\nabla \times (\mu \vec{H}). \tag{68}$$

If it is assumed that the permeability of the medium is constant with respect to the space coordinates, equation (68) becomes

$$\nabla \times \nabla \times \vec{E} = -i\omega\mu\nabla \times \vec{H}. \tag{69}$$

The permeability is then taken to be that of free space, μ_o.

Upon substitution of (o7) into (69), we have

$$\nabla \times \nabla \times \vec{E} = -i\omega\mu(\sigma + i\omega\epsilon)\vec{E},$$

or

$$\nabla \times \nabla \times \vec{E} = k^2\vec{E}, \tag{70}$$

where

$$k^2 = \omega^2\mu\epsilon - i\omega\mu\sigma.$$

If the charge density in the region of interest is taken to be negligible, equation (70) becomes the familiar vector Helmholtz equation

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \tag{71}$$

Except for the simplest of cases, the vector Helmholtz equation cannot be resolved into three scalar equations which are easily solved. For certain coordinate systems, however, there exist three independent vector solutions which are designated as $\vec{L}$, $\vec{M}$ and $\vec{N}$ type solutions.[6, 9]

If n is a constant vector with certain properties, then the three solutions are

$$\vec{L} = \nabla \psi \tag{72}$$

$$\vec{M} = \nabla \times \vec{n} \psi \tag{73}$$

$$\vec{N} = \frac{1}{k} \nabla \times \nabla \times \vec{n} \psi , \tag{74}$$

where ψ is a function satisfying the scalar equation

$$\nabla^2 \psi + k^2 \psi = 0 \tag{75}$$

in the particular coordinate system being used.

If we let $\vec{n} = \vec{r} = r \vec{a}_r$, the radial vector for the spherical system, then

$$\vec{M} = \nabla \times \vec{a}_r (r \psi)$$

$$\vec{N} = \frac{1}{k} \nabla \times \nabla \times \vec{a}_r (r \psi) .$$

M, when decomposed into its three components, is

$$M_r = 0$$

$$M_\theta = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$M_\phi = -\frac{\partial\psi}{\partial\theta}\ .$$

Likewise,

$$(\nabla \times \vec{M})_r = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) - \frac{1}{r\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$(\nabla \times \vec{M})_\theta = \frac{\partial}{\partial\theta}(\frac{\partial\psi}{\partial r} + \frac{1}{r}\psi)$$

$$(\nabla \times \vec{M})_\phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\frac{\partial\psi}{\partial r} + \frac{1}{r}\psi)\ .$$

If the vector operations of

$$\nabla \times \nabla \times \vec{M} - k^2\vec{M} = 0$$

are carried out, the results are

$$(\nabla \times \nabla \times \vec{M})_r - k^2 M_r = 0$$

$$(\nabla \times \nabla \times \vec{M})_\theta - k^2 M_\theta = -\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta})\right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{k^2}{\sin\theta}\frac{\partial}{\partial\phi}(\psi)$$

$$(\nabla \times \nabla \times \vec{M})_\phi - k^2 M_\phi = \frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta})\right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + k^2\frac{\partial}{\partial\theta}(\psi)\ .$$

But the expression in each of the brackets is just $\nabla^2\psi$ in the spherical coordinates. Therefore,

$$(\nabla \times \nabla \times \vec{M})_\theta - k^2 M_\theta = -\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\nabla^2\psi + k^2\psi\right] = 0$$

$$(\nabla \times \nabla \times \vec{M})_\phi - k^2 M = \frac{\partial}{\partial\theta}\left[\nabla^2\psi + k^2\psi\right] = 0 .$$

Thus, since the ψ scalar is constructed to satisfy equation (75), the vector $\vec{M}$ is a solution of the vector equation (71). In a similar fashion $\vec{N}$ may also be shown to be a solution of (71).

The electric field may now be represented as a $\vec{M}$ or $\vec{N}$ type field. It was shown that $\vec{M}$ contains only components transverse to the radial direction, while $\vec{N}$ has not only transverse components, but also a radial component. If the electric field is described by $\vec{E} = \vec{M}$, $\vec{H}$ becomes

$$\vec{H} = -\frac{k}{i\omega\mu}\nabla \times \vec{M},$$

and the system of fields is transverse electric in nature. If $\vec{E} = \vec{N}$, then the fields are seen to be transverse magnetic.

B. Derivation of the ψ-Function

For the spherical system under consideration, the expansion of the scalar equation (75) is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0. \quad (76)$$

If we assume a solution of the form

$$\psi(r, \theta, \phi) = \psi_r(r)\,\psi_\theta(\theta)\,\psi_\phi(\phi), \tag{77}$$

we have, after substitution of (77) into (76),

$$\frac{1}{\psi_r}\frac{d}{dr}\left(r^2\frac{d\psi_r}{dr}\right) + \frac{1}{\psi_\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_\theta}{d\theta}\right)$$

$$+ \frac{1}{\psi_\phi}\frac{1}{\sin^2\theta}\frac{d^2\psi_\phi}{d\phi^2} + k^2r^2 = 0. \tag{78}$$

Because of the separated state of the variables, three equations are extracted from equation (78); namely,

$$\frac{d^2\psi_\phi}{d\phi^2} = -m^2\psi_\phi \tag{79}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_\theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\psi_\theta + p^2\psi_\theta = 0 \tag{80}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_r}{dr}\right) + \left(k^2 - \frac{p^2}{r^2}\right)\psi_r = 0, \tag{81}$$

where m^2 and p^2 are separation constants. Equation (80), when expanded, becomes

$$\frac{d^2\psi_\theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\psi_\theta}{d\theta} + \left(p^2 - \frac{m^2}{\sin^2\theta}\right)\psi_\theta = 0.$$

After the substitution of $\lambda = \cos\theta$, this becomes

$$(1-\lambda^2)\frac{d^2\psi_\theta}{d\lambda^2} - 2\lambda\frac{d\psi_\theta}{d\lambda} + \left(p^2 - \frac{m^2}{1-\lambda^2}\right)\psi_\theta = 0, \tag{82}$$

an equation having singularities at $\lambda = \pm 1$. If $p^2 = n(n+1)$, the solution of interest will remain finite at these singularities.

The solutions to equations (79) are in the form of sinusoids. In order that the ψ-solutions be single-valued, the constant m must take on integer values, $m = 0, \pm 1, \pm 2, \ldots$. The solution to (79) may then be written as

$$\psi_\phi = a \cos m\phi + b \sin m\phi ,$$

or

$$\psi_\phi = a e^{im\phi} + b e^{-im\phi}. \tag{83}$$

The second form is used in this problem since this is the variational form assumed in the finite-difference approach. If tne m in equation (82) takes on the value $m = 0$, there follows

$$(1-\lambda^2)\frac{d^2\psi_\theta}{d\lambda^2} - 2\lambda\frac{d\psi_\theta}{d\lambda} + n(n+1)\psi_\theta = 0. \tag{84}$$

Differentiation of (84) with respect to λ yields

$$(1-\lambda^2)\frac{d^2}{d\lambda^2}\left(\frac{d\psi_\theta}{d\lambda}\right) - 4\lambda\frac{d^2\psi_\theta}{d\lambda^2} - 2\frac{d\psi_\theta}{d\lambda} + n(n+1)\frac{d\psi_\theta}{d\lambda} = 0 .$$

A second differentiation yields

$$(1-\lambda^2)\frac{d^2}{d\lambda^2}\left(\frac{d^2\psi_\theta}{d\lambda^2}\right) - 2(2+1)\lambda\frac{d}{d\lambda}\left(\frac{d^2\psi_\theta}{d\lambda^2}\right) + [n(n+1) - 2(2+1)]\frac{d^2\psi_\theta}{d\lambda^2} = 0,$$

and the third time,

$$(1-\lambda^2)\frac{d^2}{d\lambda^2}\left(\frac{d^3\psi_\theta}{d\lambda^3}\right) - 2(3+1)\lambda\frac{d}{d\lambda}\left(\frac{d^3\psi_\theta}{d\lambda^3}\right) + [n(n+1) - 3(3+1)]\frac{d^3\psi_\theta}{d\lambda^3} = 0.$$

The general expression for the m^th derivative of equation (84) is

$$(1-\lambda^2)\frac{d^2}{d\lambda^2}\left(\frac{d^m\psi_\theta}{d\lambda^m}\right) - 2(m+1)\lambda\frac{d}{d\lambda}\left(\frac{d^m\psi_\theta}{d\lambda^m}\right)$$

$$+ [n(n+1) - m(m+1)]\frac{d^m\psi_\theta}{d\lambda^m} = 0.$$

Now, letting

$$v = (1-\lambda^2)^{m/2},$$

equation (86) becomes

$$(1-\lambda^2)\frac{d^2}{d\lambda^2}[(1-\lambda^2)^{-m/2}v] - 2(m+1)\lambda\frac{d}{d\lambda}[(1-\lambda^2)^{-m/2}v]$$

$$+ [n(n+1) - m(m+1)](1-\lambda^2)^{-m/2}v = 0. \qquad (87)$$

After a small amount of algebraic manipulation, one finds that (87) reduces to

$$(1-\lambda^2)\frac{d^2v}{d\lambda^2} - 2\lambda\frac{dv}{d\lambda} + [n(n+1) - \frac{m^2}{1-\lambda^2}]v = 0. \qquad (88)$$

Equation (84) is a special equation which has solutions in the form of the Legendre polynomial $P_n(\lambda)$, so the solution to (88) must be

$$\psi_\theta = c[(1-\lambda^2)^{m/2}\frac{d^m}{d\lambda^m}P_n(\lambda)].$$

The function within the bracket is called the associated Legendre Polynomial of order m and degree n. It is symbolically noted as

$$\psi_\theta = c\,P_n^m(\lambda). \tag{89}$$

The third of the three separated equations is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_r}{dr}\right) + \left[k^2 - \frac{n(n+1)}{r^2}\right]\psi_r = 0.$$

The substitution of $\rho = kr$ and $R = \psi_r r$ leads to

$$\frac{d^2R}{d\rho^2} + \left[1 - \frac{n(n+1)}{\rho^2}\right]R = 0. \tag{90}$$

Now, if $R = \sqrt{\rho}\,S$, equation (90) becomes Bessel's equation

$$\frac{d^2S}{d\rho^2} + \frac{1}{\rho}\frac{dS}{d\rho} + \left[1 - \frac{(n+1/2)^2}{\rho^2}\right]S = 0,$$

with solutions of the form of

$$S = Z_{n+1/2}(\rho),$$

where $Z_{n+1/2}(\rho)$ represents the various Bessel functions of order $n+1/2$. The solution to the separated equation in (81) is therefore

$$\psi_r = d\,\frac{1}{\sqrt{r}}\,Z_{n+1/2}(\rho).$$

The Hankel functions are chosen for this problem, and so the ψ_r may be written in terms of these Bessel functions of the third kind,

$$\psi_r = d\,\frac{1}{\sqrt{r}}\,H^{(1)}_{n+1/2}(kr) + e\,\frac{1}{\sqrt{r}}\,H^{(2)}_{n+1/2}(kr),$$

and, using the definitions found in Appendix, the final form of the radial component of the separated solutions is

$$\psi_r = f h_n^{(1)}(kr) + g h_n^{(2)}(kr).$$

The combination of equations (83), (89) and (91) yields the total form of the solution to the scalar wave equation (75) in spherical coordinates,

$$\psi = [a e^{im\phi} + b e^{-im\phi}][c P_n^m(\lambda)][f h_n^{(1)}(kr) + g h_n^{(2)}(kr)].$$

Two of these terms may be discarded for this problem. Since a dependence on ϕ of the fields for the finite-difference calculations was taken to be $e^{im\phi}$, the term $e^{-im\phi}$ is not required here.

The selection of the proper Hankel function depends on the fact that only an outgoing wave is desired. Now,

$$k^2 = \omega^2 \mu\epsilon - i\omega\mu\sigma .$$

For a passive medium, σ is positive real so that k^2 will lie in the fourth, and, for the particular root chosen, k will have a negative imaginary part. In order for the wave to die off at very large distance, the second spherical Hankel function, $h_n^{(2)}(kr)$, must be selected for the representation of the outward propagating wave.

The general solution is therefore

$$\psi = a e^{im\phi} h_n^{(2)}(kr) P_n^m(\lambda) \tag{91}$$

where the constant a is a combination of previous constants.

C. The Final Form of the Fields

The $\vec{M}$ and $\vec{N}$ solutions may now be constructed using (91).

$$M_r = 0$$

$$M_\theta = \frac{im}{\sin\theta}\, e^{im\phi}\, h_n^{(2)}(kr)\, P_n^m(\lambda)$$

$$M_\phi = -\, e^{im\phi}\, h_n^{(2)}(kr)\, \frac{dP_n^m(\lambda)}{d\theta}$$

$$N_r = \frac{n(n+1)}{kr}\, e^{im\phi}\, h_n^{(2)}(kr)\, P_n^m(\lambda)$$

$$N_\theta = \frac{1}{kr}\, e^{im\phi}\, \frac{d}{dr}\left[r\, h_n^{(2)}(kr)\right] \frac{dP_n^m(\lambda)}{d\theta}$$

$$N_\phi = \frac{im}{kr\sin\theta}\, e^{im\phi}\, \frac{d}{dr}\left[r\, h_n^{(2)}(kr)\right] P_n^m(\lambda).$$

When $\lambda = \cos\theta$ is substituted into these six equations, $\vec{M}$ and $\vec{N}$ become

$$M_r = 0 \tag{92}$$

$$M_\theta = \frac{im}{\sqrt{1-\lambda^2}}\, e^{im\phi}\, h_n^{(2)}(kr)\, P_n^m(\lambda) \tag{93}$$

$$M_\phi = \sqrt{1-\lambda^2}\; e^{im\phi}\, h_n^{(2)}(kr)\, \frac{dP_n^m(\lambda)}{d\lambda} \tag{94}$$

$$N_r = \frac{n(n+1)}{kr}\, e^{im\phi}\, h_n^{(2)}(kr)\, P_n^m(\lambda) \tag{95}$$

$$N_\theta = \frac{\sqrt{1-\lambda^2}}{kr}\, e^{im\phi}\, \frac{d}{dr}\left[r\, h_n^{(2)}(kr)\right] \frac{dP_n^m(\lambda)}{d\lambda} \tag{96}$$

$$N_\phi = \frac{im}{kr\sqrt{1-\lambda^2}}\, e^{im\phi}\, \frac{d}{dr}\left[r\, h_n^{(2)}(kr)\right] P_n^m(\lambda). \tag{97}$$

Since any solenoidal $\vec{E}$ field may be constructed from a linear sum of transverse electric and transverse magnetic waves,

$$E = \sum_n (A_n \vec{M}_n + B_n \vec{N}_n), \tag{98}$$

where A_n and B_n are constants determined by the boundary conditions. In other words, A_n is the amplitude of the n^{th} mode transverse electric wave, and B_n is the amplitude of the n^{th} mode transverse magnetic wave. The index n must be summed up to a value high enough to include the highest ordered mode present in the system. Since the finite-difference method employed N points in the angular direction, the highest mode we shall consider is the N^{th} mode.

The magnetic field $\vec{H}$, as derived from Maxwell's equation, is

$$\vec{H} = -\frac{1}{i\omega\mu} \nabla \times \left\{ \sum_n (A_n \vec{M}_n + B_n \vec{N}_n) \right\},$$

and, since A_n and B_n are constant with respect to the space coordinates,

$$\vec{H} = -\frac{1}{i\omega\mu} \sum_n \left\{ A_n (\nabla \times \vec{M}_n) + B_n (\nabla \times \vec{N}_n) \right\}. \tag{99}$$

Equation (74) defining $\vec{N}$ is rewritten, yielding

$$\vec{N} = \frac{1}{k} \nabla \times \vec{M}, \tag{100}$$

and from equation (73), $\vec{M}$ is given by

$$\vec{M} = \nabla \times \vec{n}\psi = -\frac{1}{k^2} \nabla \times \vec{n}(-k^2 \psi)$$

$$\vec{M} = -\frac{1}{k^2} \nabla \times (\vec{n}\nabla^2 \psi) . \tag{101}$$

From a vector identity and the knowledge of the properties of the vector $\vec{n}$, we have

$$\nabla \times (\nabla\psi \times \vec{n}) = -n\nabla \cdot \nabla\psi = -\vec{n}\nabla^2 \psi . \tag{102}$$

Upon substitution of (102) into equation (101), the vector solution $\vec{M}$ becomes

$$\vec{M} = \frac{1}{k^2} \nabla \times \nabla \times (\nabla\psi \times \vec{n}) .$$

It can be seen that

$$\vec{M} = \nabla\psi \times \vec{n},$$

if equation (73) is expanded, and therefore,

$$\vec{M} = \frac{1}{k^2} \nabla \times \nabla \times \vec{M} ,$$

which, by equation (100), reduces finally to

$$\vec{M} = \frac{1}{k} \nabla \times \vec{N} .$$

The magnetic field is thus given most conveniently by

$$\vec{H} = -\frac{k}{i\omega\mu} \sum_n [A_n\vec{N}_n + B_n\vec{M}_n] . \tag{103}$$

Finally, with equations (92) through (97) substituted into (98) and (103), the general field expressions for the spherical wave propagating outwardly are

$$E_r = \sum_{n=1}^{N} \left\{ G_n \frac{n(n+1)}{kr} e^{im\phi} h_n^{(2)}(kr) P_n^m(\lambda) \right\} \tag{104}$$

$$E_\theta = \sum_{n=1}^{N} \left\{ F_n \frac{im}{\sqrt{1-\lambda^2}} e^{im\phi} h_n^{(2)}(kr) P_n^m(\lambda) \right.$$

$$\left. - G_n \frac{\sqrt{1-\lambda^2}}{kr} e^{im\phi} \frac{d}{dr} [r h_n^{(2)}(kr)] \frac{dP_n^m(\lambda)}{d\lambda} \right\} \tag{105}$$

$$E_\phi = \sum_{n=1}^{N} \left\{ F_n \sqrt{1-\lambda^2} e^{im\phi} h_n^{(2)}(kr) \frac{dP_n^m(\lambda)}{d\lambda} \right.$$

$$\left. + G_n \frac{im}{kr\sqrt{1-\lambda^2}} e^{im\phi} \frac{d}{dr} [r h_n^{(2)}(kr)] P_n^m(\lambda) \right\} \tag{106}$$

$$H_r = \sum_{n=1}^{N} \left\{ -F_n \frac{n(n+1)}{i\omega\mu r} e^{im\phi} h_n^{(2)}(kr) P_n^m(\lambda) \right\} \tag{107}$$

$$H_\theta = \sum_{n=1}^{N} \left\{ F_n \frac{\sqrt{1-\lambda^2}}{i\omega\mu r} e^{im\phi} \frac{d}{dr} [r h_n^{(2)}(kr)] \frac{dP_n^m(\lambda)}{d\lambda} \right.$$

$$\left. - G_n \frac{mk}{\omega\mu\sqrt{1-\lambda^2}} e^{im\phi} h_n^{(2)}(kr) P_n^m(\lambda) \right\} \tag{108}$$

$$H_\phi = \sum_{n=1}^{N} \left\{ -F_n \frac{m}{\omega\mu r\sqrt{1-\lambda^2}} e^{im\phi} \frac{d}{dr} [r h_n^{(2)}(kr)] P_n^m(\lambda) \right.$$

$$\left. - G_n \frac{\sqrt{1-\lambda^2}}{i\omega\mu} e^{im\phi} h_n^{(2)}(kr) \frac{dP_n^m(\lambda)}{d\lambda} \right\} . \tag{109}$$

The four expressions of these equations for the tangential fields may be written in symbolic form using notation introduced previously.

$$E_\theta(M, q) = \sum_{n=1}^{N} [F_n a_{qn} + G_n a'_{qn}] \tag{110}$$

$$E_\phi(M, q) = \sum_{n=1}^{N} [F_n b_{qn} + G_n b'_{qn}] \tag{111}$$

$$H_\theta(M, q) = \sum_{n=1}^{N} [F_n f_{qn} + G_n f'_{qn}] \tag{112}$$

$$H_\phi(M, q) = \sum_{n=1}^{N} [F_n g_{qn} + G_n g'_{qn}], \tag{113}$$

where the terms a_{qn}, a'_{qn}, b_{qn}, b'_{qn}, f_{qn}, f'_{qn}, g_{qn} and g'_{qn} are the complex terms in the field equations (105), (106), (108), and (109) and are functions of both the mode number and the angular position.

IV. COMBINATION OF THE EQUATIONS AND BOUNDARY CONDITIONS

There now exist two expressions for each of the fields tangential to the outer boundary. Equations (62) through (65) are equated to (110) through (113), yielding

$$\sum_{n=1}^{N} [a_{qn} F_n + a'_{qn} G_n] = \sum_{n=1}^{N} [A_{qn} H_\theta(1, n) + A'_{qn} H_\phi(1, n)]$$
$$+ \sum_{n=1}^{N} [\alpha_{qn} J_{s\phi}(n) + \alpha'_{qn} J_{s\phi}(n) + \beta_{qn} \rho_s(n)] \quad (115)$$

$$\sum_{n=1}^{N} [b_{qn} F_n + b'_{qn} G_n] = \sum_{n=1}^{N} [B_{qn} H_\theta(1, n) + B'_{qn} H_\phi(1, n)]$$
$$+ \sum_{n=1}^{N} [\gamma_{qn} J_{s\phi}(n) + \gamma'_{qn} J_{s\theta}(n) + \delta_{qn} \rho_s(n)] \quad (116)$$

$$\sum_{n=1}^{N} [f_{qn} F_n + f'_{qn} G_n] = \sum_{n=1}^{N} [C_{qn} H_\theta(1, n) + C'_{qn} H_\phi(1, n)]$$
$$+ \sum_{n=1}^{N} [\xi_{qn} J_{s\phi}(n) + \xi'_{qn} J_{s\phi}(n) + \eta_{qn} \rho_s(n)] \quad (117)$$

$$\sum_{n=1}^{N} [g_{qn} F_n + g'_{qn} G_n] = \sum_{n=1}^{N} [D_{qn} H_\theta(1, n) + D'_{qn} H_\phi(1, n)]$$
$$+ \sum_{n=1}^{N} [\zeta_{qn} J_{s\phi}(n) + S_{qn} J_{s\theta}(n) + \nu_{qn} \rho_s(n)] \quad (118)$$

The quantities which are known in these 4 N equations are all the coefficients plus the source terms $J_{s\phi}(n)$, $J_{s\theta}(n)$ and $\rho_s(n)$ while the unknowns

are $H_\theta(1,n)$, $H_\phi(1,n)$, F_n and G_n. There are, therefore, 4 N equations with 4 N unknowns.

A. Evaluation of the Fields at the Inner Boundary

Since, for each q, both the coefficients α_{qn}, α'_{qn}, β_{qn}, γ_{qn}, γ'_{qn}, δ_{qn}, ξ_{qn}, ξ'_{qn}, η_{qn}, ζ_{qn}, ζ'_{qn} and ν_{qn} and the sources $J_{s\phi}(n)$, $J_{s\theta}(n)$ and $\rho_s(n)$ are known in the problem, let the following substitutions be made

$$S_{1q} = -\sum_{n=1}^{N} [\alpha_{qn} J_{s\phi}(n) + \alpha'_{qn} J_{s\theta}(n) + \beta_{qn}\rho_s(n)]$$

$$S_{2q} = -\sum_{n=1}^{N} [\gamma_{qn} J_{s\phi}(n) + \gamma'_{qn} J_{s\theta}(n) + \delta_{qn}\rho_s(n)]$$

$$S_{3q} = -\sum_{n=1}^{N} [\xi_{qn} J_{s\phi}(n) + \xi'_{qn} J_{s\theta}(n) + \eta_{qn}\rho_s(n)]$$

$$S_{4q} = -\sum_{n=1}^{N} [\zeta_{qn} J_{s\phi}(n) + \zeta'_{qn} J_{s\theta}(n) + \nu_{qn}\rho_s(n)]$$

Then equations (114) through (117) may be rewritten in matrix form (where the coefficients a_{qn}, a'_{qn}, b_{qn}, f_{qn}, f'_{qn}, g_{qn} and g'_{qn} are the negative values of those calculated previously). This matrix is shown on the next page.

$$
\underset{\Delta_2}{\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1N} & A'_{11} & \cdots & A'_{1N} & a_{11} & \cdots & a_{1N} & a'_{11} & \cdots & a'_{1N} \\
A_{21} & A_{22} & \cdots & A_{2N} & A'_{21} & \cdots & A'_{2N} & a_{21} & \cdots & a_{2N} & a'_{21} & \cdots & a'_{2N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
A_{N1} & A_{N2} & \cdots & A_{NN} & A'_{N1} & \cdots & A'_{NN} & a_{N1} & \cdots & a_{NN} & a'_{N1} & \cdots & a'_{NN} \\
B_{11} & B_{12} & \cdots & B_{1N} & B'_{11} & \cdots & B'_{1N} & b_{11} & \cdots & b_{1N} & b'_{11} & \cdots & b'_{1N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
B_{N1} & B_{N2} & \cdots & B_{NN} & B'_{N1} & \cdots & B'_{NN} & b_{N1} & \cdots & b_{NN} & b'_{N1} & \cdots & b'_{NN} \\
C_{11} & C_{12} & \cdots & C_{1N} & C'_{11} & \cdots & C'_{1N} & f_{11} & \cdots & f_{1N} & f'_{11} & \cdots & f'_{1N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NN} & C'_{N1} & \cdots & C'_{NN} & f_{N1} & \cdots & f_{NN} & f'_{N1} & \cdots & f'_{NN} \\
D_{11} & D_{12} & \cdots & D_{1N} & D'_{11} & \cdots & D'_{1N} & g_{11} & \cdots & g_{1N} & g'_{11} & \cdots & g'_{1N} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
D_{N1} & D_{N2} & \cdots & D_{NN} & D'_{N1} & \cdots & D'_{NN} & g_{N1} & \cdots & g_{NN} & g'_{N1} & \cdots & g'_{NN}
\end{bmatrix}}
\underset{\Delta_3}{\begin{bmatrix}
H_\theta(1,1) \\
H_\theta(1,2) \\
\vdots \\
H_\theta(1,N) \\
H_\phi(1,1) \\
\vdots \\
H_\phi(1,N) \\
F_1 \\
\vdots \\
F_N \\
G_1 \\
\vdots \\
G_N
\end{bmatrix}}
=
\begin{bmatrix}
S_{11} \\
S_{12} \\
\vdots \\
S_{1N} \\
S_{21} \\
\vdots \\
S_{2N} \\
S_{31} \\
\vdots \\
S_{3N} \\
S_{41} \\
\vdots \\
S_{4N}
\end{bmatrix}
$$

The system of matrices may now be inverted by some method to give the values of $H_\theta(l, q)$ and $H_\phi(l, q)$ for each q, $q = 1, 2, \ldots, N$. Once this is done, all the tangential fields at the inner boundary are determined, and the problem may then be concluded.

B. Field Calculations

Once the tangential fields at the inner boundary are completely determined, all fields within the system are calculated from the finite-difference equations (38) through (43) in a step-by-step manner as before. The proper boundary conditions are applied at the discontinuity, and the procedure is continued until all six field components are known at each of the grid points. This concludes the procedure except for perhaps a few checks on the accuracy of the method.

Although checks would increase the length of the computation it might be desirable, at least until the calculations prove to be reliable, to perform one or more checks on the accuracy of the calculations.

One such verification consists of solving for the coefficients F_n and G_n in the matrix which was to be inverted. The fields at the outer boundary calculated from those coefficients in equations (104) through (109) should be the same as those calculated in the last iteration process.

Any computational instability of the fields either in the radial direction or polar angle direction must be carefully examined for computational troubles such as significant figure problems.

V. COMPUTATIONAL METHODS

A program for the digital computer may be most conveniently written with several subprograms or subroutines. A brief outline of such a program follows.

Quantities which must be inserted into the computer initially include the frequency, ω; the plasma frequency, ω_p; the number of grid points in the radial direction, M; the number of grid points desired for λ, N; the molecular weight of ions or mass ratio of ions to electrons, a_o; the radius of the inner boundary, R_{min}; the radius of the outer boundary, R_{max}; the current distributions, $J_\theta(q)$ and $J_\phi(q)$; the charge density, $\rho(q)$; and the radial location of the sources. Of course, constants such as μ_o and π must be included in the program. The quantities listed above are read into the computer to enable the programmer to change one or more of these parameters easily.

After all inputs are within the computer, the conductivity matrix is calculated in a separate subroutine. This matrix has components which are found in Appendix B. These numbers are stored for later use.

The next quantities to be calculated in another subroutine are the coefficients of the matrix (66). The functions a_{qn}, a'_{qn}, b_{qn}, b'_{qn}, f_{qn}, f'_{qn}, g_{qn}, and g'_{qn} of equations (110) through (113) are then computed with separate subroutines for the calculation of the Hankel functions and the associated Legendre polynomials. Once all elements in the first matrix

Δ_2 are determined, the matrix is inverted by Gaussian elimination for the field quantities $H_\theta(l, q)$ and $H_\phi(l, q)$; this may also be accomplished in a separate subroutine.

The main program may then be used for the final calculation of all the field quantities. To eliminate the need for even more storage space, the six field components are printed as they are calculated at each radial level.

Although the use of many subroutines increases the amount of storage space required, it greatly simplifies the isolation of errors or possible trouble areas in the program, as well as simplifying the writing of the program. If storage space is the prime factor in the calculations, as it is likely to be, all calculations may be accomplished in the main program. If further reduction in either the amount of core storage or time needed is necessary, the problem may be solved in separate parts or programs with the output of one program used as the input of the next. In such a way the conductivities, the coefficients for the matrix (66), and the eight functions of (110) through (113) may be calculated separately and used as input to a final program to invert the matrix Δ_2.

A high-speed digital computer possessing an extremely large memory capacity is needed for quantitative results from this problem. Because of the large storage volume needed, all calculation should, if possible, be accomplished using only single precision arithmetic. For this reason means must be found to circumvent any significant figure problems.

Several subtle difficulties are present in the direct application of some of the difference equations. One such problem of significant figures lies in the calculation of H_ϕ from equation (30). For this problem an auxiliary field E_p was utilized by Chapman (1966) as defined by

$$E_p = k_r E_r + k_\theta E_\theta \, .$$

For further details see reference 3.

VI. CONCLUSIONS

Many notions present themselves as to how the quantitative results may be used. One aspect of the problem that certainly should be examined is the effect of various current distributions and charge density distributions on the fields at the surface of the earth. A person need not limit the source to one current sheet, but, by extending the idea of this problem, he might examine the fields with several such current rings or sheets of charge encircling the earth.

The variation of the electromagnetic field with the several parameters of interest is another needed route of investigation. These parameters include, among others, the frequency, plasma frequency, altitude of the prevailing sources, and altitude of the undistorted dipole field cavity.

In order that an adequate resolution may be achieved and a minimum chance of instability due to computational errors be realized, a large number of grid points must be involved in the quantitative consideration of this problem. For this reason the present digital computer available (Control Data Corporation 1604) could only be utilized if the problem were divided into the several parts mentioned previously. Attempts in the computational region will most likely be made after the installation of the new computer (Control Data Corporation 6600) in the Computation Center at The University of Texas.

APPENDIX A

FINITE-DIFFERENCE FORMULAE

The finite-difference formulae are based on polynomial approximations. See reference 4 for details of the formulae. Three methods of differencing exist; namely, forward differencing, backward differencing, and central differencing. For the use of this problem, only the expressions for the derivatives of functions are of interest.

Let y(x) be a function of x (it may also be a function of other variables, but each of these is held constant for these approximations of the partial derivatives, and so no generality is lost by the consideration of only x), y_n be the value of the function y at the value of $x = y_n$, y_{n+1} be the value of the function at the next point chosen on the x-axis ($x_{n+1} > x_n$), and y_{n-1} be the value of y at the point directly to the left of x_n on the x-axis. The figure illustrates the notation.

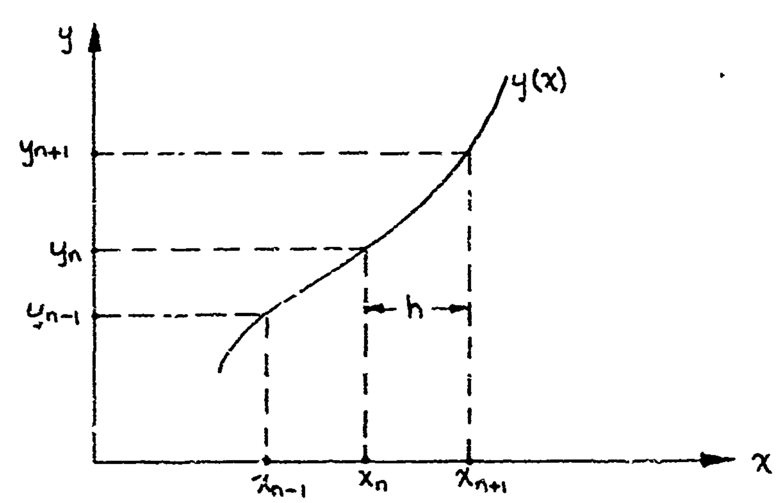

Figure 4

Definitions for the Finite-Difference Formulae

The forward difference formula for the derivative of y with respect to x evaluated at $x = x_n$ is given by the series

$$\frac{dy_n}{dx} = \frac{1}{h}\left[(y_{n+1} - y_n) - \frac{1}{2}(y_{n+2} - 2y_{n+1} + y_n) + \cdots\right],$$

where h is the increment on x between x_n and x_{n+1}. The backward difference formula is, if h is also the increment between x_{n-1} and x_n,

$$\frac{dy_n}{dx} = \frac{1}{h}\left[y_n - y_{n-1}) + \frac{1}{2}(y_n - 2y_{n-1} + y_{n-2}) - \cdots\right],$$

and the formula for the more accurate central difference is

$$\frac{dy_n}{dx} = \frac{1}{2h}\left[(y_{n+1} - y_{n-1}) - \frac{1}{6}(y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}) + \cdots\right].$$

If approximations to the first order only are used, the difference formulae become simply

$$\frac{dy_n}{dx} = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}$$

for the forward difference,

$$\frac{dy_n}{dx} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

for the backward difference, and

$$\frac{dy_n}{dx} = \frac{y_{n+1} - y_{n-1}}{2(x_{n+1} - x_n)}$$

for the central difference. The central difference formula is most convenient when the increments taken in the x-direction are all equal. However, non-uniform increments may also be used, with the various formulae being modified accordingly.

The central difference formula is used whenever possible, and so it is employed in the computation of the derivatives of the field with respect to λ at all points. The central difference is used to approximate the derivatives with respect to r except in the initial step from the inner boundary when the forward difference is used.

Let $\Delta\lambda$ be the incremental change in λ used in the grid, and Δr be the radial distance between two adjacent levels. Also, the notation A(p, q) is taken to mean the value of A at the radius r_p and angle (cosine)λ. A(p+1, q) therefore signifies the value of A at the next radial and the same value of λ. The difference formulae are then

$$\frac{dA}{dr} = \frac{A(p+1, q) - A(p, q)}{\Delta r}$$

$$\frac{dA}{dr} = \frac{A(p, q) - A(p-1, q)}{\Delta r}$$

$$\frac{dA}{dr} = \frac{A(p+1, q) - A(p-1, q)}{2\Delta r}$$

for the forward, backward, and central difference, respectively, for the radial derivative, and

$$\frac{dA}{d\lambda} = \frac{A(p, q+1) - A(p, q)}{\Delta\lambda}$$

$$\frac{dA}{d\lambda} = \frac{A(p, q) - A(p, q-1)}{\Delta\lambda}$$

$$\frac{dA}{d\lambda} = \frac{A(p, q+1) - A(p, q-1)}{2\Delta\lambda}$$

for the same differences of the tangential derivatives. All six of these expressions are for the derivatives evaluated at the point (p, q).

APPENDIX B

DERIVATION OF THE CONDUCTIVITY MATRIX

Bostick (1964) derived the conductivity tensor from considerations of particle interaction using basic equations of plasma physics. The complex conductivity tensor is based on the generalized Ohm's law, the equation of motion for the charged particles in an ionized gas, and the equation of motion for neutral particles. The following is an outline of the derivation of the conductivity elements used by Chapman (1966). It is presented here since the conductivities must be calculated for this problem.

Let f_1, f_2 and f_3 be defined by

$$f_1 = f_{en} - \frac{f_{in}}{2} \qquad \text{(A-1)}$$

$$f_2 = \frac{f_{in}}{2} + \frac{m_e}{m_i} f_{en} \qquad \text{(A-2)}$$

$$f_3 = f_{ie} + f_{en} + \frac{m_e}{m_i} f_{in} , \qquad \text{(A-3)}$$

where the terms f_{en}, f_{in} and f_{ie} are the effective collision frequencies between electrons and neutral particles, ions and neutral particles, and ions and electrons, respectively, and m_e and m_i are the electron and ion masses.

If the electron plasma frequency ω_p is defined as

$$\omega_p^2 = \frac{N_o q^2}{m_e \varepsilon_o}$$

where N_o is the electron charge density, then the generalized Ohm's law is given by

$$\frac{1}{\varepsilon_o \omega_p^2}\left[(i\omega + f_3)\vec{J} + \omega_e \vec{J} \times \vec{k}\right] = \vec{E} + \vec{V}_p \times \vec{k} B_o$$

$$+ \frac{B_o f_1}{\omega_e}(\vec{V}_p - \vec{V}_n) + \frac{\Delta p_e}{N_o q}. \qquad \text{(A-4)}$$

In equation (A-4), $\vec{J}$ is the current desnity, k is the unit vector in the +z direction, ω_e is the cyclotron resonance frequency for electrons, $\vec{V}_p$ is the average plasma velocity, $\vec{V}_n$ is the average velocity of the neutral particles, and p_e is the perturbation in pressure of electrons.

The equation of motion for the charged component of the plasma is

$$i\omega N_o m \vec{V}_p = \frac{B_o f_i}{\omega_e}\vec{J} - N_o m f_2(\vec{V}_p - \vec{V}_n) + \vec{J} \times \vec{k} B_o - \Delta p_p \qquad \text{(A-5)}$$

in which m is the ion mass and p_p is the perturbation in pressure difference of ions and electrons.

The equation of motion for the neutral particles is

$$i\omega N m \vec{V}_n = -\frac{B_o f_1}{\omega_e}\vec{J} - N_o m f_2(\vec{V}_n - \vec{V}_p) - \Delta p_n \qquad \text{(A-6)}$$

where N is the particle density of neutral particles, and p_n is the perturbed pressure of the neutral particles.

Equation (A-4) is rearranged to give

$$(i\omega N_o m_i + N_o m_i f_2)\vec{V}_p - N_o m_i f_2 \vec{V}_n = \frac{B_o f_1}{\omega_e}\vec{J} + \vec{J}\times\vec{k}B_o \qquad \text{(A-7)}$$

and equation (A-6) is rearranged to give

$$-N_o m_i f_2 \vec{V}_p + (N_o m_i f_2 + i\omega N m_i)\vec{V}_n = -\frac{B_o f_1}{\omega_e}\vec{J}\,. \qquad \text{(A-8)}$$

If equations (A-7) and (A-8) are solved simultaneously for $\vec{V}_n$ and $\vec{V}_p$, the results are

$$\vec{V}_p = \frac{(f_2 + i\frac{N}{N_o})\left(\frac{B_o f_1}{\omega_e}\vec{J} + \vec{J}\times\vec{k}B_o\right) - \frac{B_o f_1 f_2}{\omega_e}\vec{J}}{N_o m_i\left[i\omega f_2(1+\frac{N}{N_o}) - \omega^2\frac{N}{N_o}\right]} \qquad \text{(A-9)}$$

$$\vec{V}_n = \frac{-\frac{B_o f_1}{\omega_e}(f_2 + i\omega)\vec{J} + f_2\left(\frac{B_o f_1}{\omega_e}\vec{J} + \vec{J}\times\vec{k}B_o\right)}{N_o m_i\left[i\omega f_2(1+\frac{N}{N_o}) - \omega^2\frac{N}{N_o}\right]}\,. \qquad \text{(A-10)}$$

Then

$$\vec{V}_p - \vec{V}_n = \frac{\frac{B_o f_1}{\omega_e} i\omega(1+\frac{N}{N_o})\vec{J} + i\omega\frac{N}{N_o}B_o\vec{J}\times\vec{k}}{N_o m_i\left[i\omega f_2(1+\frac{N}{N_o}) - \omega^2\frac{N}{N_o}\right]}\,. \qquad \text{(A-11)}$$

When equations (A-9) and (A-11) are substituted into equation (A-4) and the resulting terms rearranged, there results

$$\vec{J}\left[\frac{i\omega+f_3}{\epsilon_o\omega_p^2} - \frac{B_o^2 f_1^2}{\omega_e^2}\,\frac{i\omega N_o m_i}{\Delta}\,(1+\frac{N}{N_o})\right]$$

$$+\,\vec{J}\times\vec{k}\left[\frac{\omega_e}{\epsilon_o\omega_p^2} - 2\,\frac{N_o m_i}{\Delta}\,\frac{B_o^2 f_1}{\omega_e}\,i\omega\frac{N}{N_o}\right] \tag{A-12}$$

$$-\,(\vec{J}\times\vec{k})\times\vec{k}\left[\frac{N_o m_i}{\Delta}\,(f_2+i\omega\frac{N}{N_o})\,B_o^2\right] = \vec{E}\,,$$

where

$$\Delta = (N_o m_i)^2\left[i\omega f_2\,(1+\frac{N}{N_o}) - \frac{\omega^2}{f_2}\,\frac{N}{N_o}\right].$$

With the cyclotron frequency defined by

$$\omega_e = \frac{qB_o}{m_e}$$

then the first term in equation (A-12) is reduced to

$$\left[\frac{i\omega+f_3}{\epsilon_o\omega_p^2} - \frac{B_o f_1^2}{\omega_e^2}\,\frac{i\omega N_o m_i}{\Delta}\,(1+\frac{N}{N_o})\right]\epsilon_o\omega_p^2 =$$

$$\left\{\frac{i\omega+f_3}{\epsilon_o\omega_p^2} - \frac{f_1}{\epsilon_o\omega_p^2}\,(\frac{m_e}{m_i})\left[\frac{f_1/f_2}{1+i\frac{\omega}{f_2}(\frac{N}{N+N_o})}\right]\right\}\epsilon_o\omega_p^2, \tag{A-13}$$

the second term becomes

$$\left[\frac{\omega_e}{\epsilon_o\omega_p^2} - 2\frac{N_o m_i}{\Delta}\frac{B_o^2 f_1}{\omega_e} i\omega\frac{N}{N_o}\right]\epsilon_o\omega_p^2 =$$

$$\left\{\frac{\omega_e}{\epsilon_o\omega_p^2} - 2\frac{\omega_i}{\epsilon_o\omega_p^2}\frac{N}{N+N_o}\left[\frac{f_1/f_2}{1+i\frac{\omega}{f_2}(\frac{N}{N+N_o})}\right]\right\}\epsilon_o\omega_p^2 , \qquad \text{(A-14)}$$

and the third term becomes

$$-\left[\frac{N_o m_i}{\Delta}(f_2 + i\omega\frac{N}{N_o})B_o^2\right]\epsilon_o\omega_p^2 =$$

$$-\frac{\omega_e^2}{i\omega\epsilon_o\omega_p^2}(\frac{m_e}{m_i})\left[\frac{1}{1+\frac{N}{N_o}}\right]\left[\frac{1+i\frac{\omega}{f_2}\frac{N}{N_c}}{1+i\frac{\omega}{f_2}\frac{N}{N+N_o}}\right]\epsilon_o\omega_p^2 \qquad \text{(A-15)}$$

Let a be defined as the right-hand side of equation (A-13), b as the right-hand side of equation (A-14), and c as the right-hand side of equation (A-15). Then

$$a\vec{J} + b\vec{J}\times\vec{k} + C(\vec{J}\times\vec{k})\times\vec{k} = \epsilon_o\omega_p^2\vec{E} . \qquad \text{(A-16)}$$

Now, the earth's magnetic field is assumed to be of a form produced by a magnetic dipole at the center of the earth with a magnetic moment of

$$M = 8.06 \times 10^{15} \text{ weber/meters.}$$

Also, from Prince, Bostick, and Smith (1964), the strength of the static field is given by

$$B_o = \frac{M}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

or

$$B_o = \frac{Mh}{r^3},$$

where

$$h = \sqrt{1+3\lambda^2}.$$

Defining $a_o = \frac{m_i}{m_e}$ and $\eta = \frac{N_o}{N}$, one finds that

$$a = (f_3 + i\omega) - \frac{f_1^2(1+\eta)}{a_o[i\omega + f_2(1+\eta)]} \tag{A-17}$$

$$b = \frac{Mqh}{m_e r^3}\left\{1 - \frac{2f_1^2}{a_o[i\omega + f_2(1+\eta)]}\right\} \tag{A-18}$$

$$c = -\left(\frac{Mqh}{m_e r^3}\right)^2\left\{\frac{\eta f_2 + i\omega}{i\omega a_o[i\omega + f_2(1+\eta)]}\right\}. \tag{A-19}$$

Equation (A-12) is specified in terms of the rectangular coordinate system, but the problem for which the conductivity matrix is needed is in the frame of the spherical coordinate system. It is therefore necessary that equation (A-12) be transformed over to the spherical system.

The unit vector $\vec{k}$ is given in terms of the unit vectors $\vec{a}_r$ and $\vec{a}_\theta$ by

$$\vec{k} = k_r \vec{a}_r + k_\theta \vec{a}_\theta,$$

where

$$k_r = -\frac{2\cos\theta}{\sqrt{1+3\cos^2\theta}} \qquad \text{(A-20)}$$

$$k_\theta = -\frac{\sin\theta}{\sqrt{1+3\cos^2\theta}} \qquad \text{(A-21)}$$

Then $\vec{J} \times \vec{k}$ becomes

$$\vec{J} \times \vec{k} = (\vec{a}_r J_r + \vec{a}_\theta J_\theta + \vec{a}_\phi J_\phi) \times (k_r \vec{a}_r + k_\theta \vec{a}_\theta)$$

$$\vec{J} \times \vec{k} = -\vec{a}_r J_\phi k_\theta + \vec{a}_\theta J_\phi k_r + \vec{a}_\phi (J_r k_\theta - J_\theta k_r), \qquad \text{(A-22)}$$

and

$$(\vec{J} \times \vec{k}) \times \vec{k} = \vec{a}_r [k_\theta (J_\theta k_r - J_r k_\theta)] + \vec{a}_\theta [k_r (J_r k_\theta - J_\theta k_r)] + \vec{a}_\phi [-J_\phi (k_r^2 + k_\theta^2)] .$$

But $k_r^2 + k_\theta^2 = 1$, so

$$(\vec{J} \times \vec{k}) \times \vec{k} = \vec{a}_r [k_\theta (J_\theta k_r - J_r k_\theta)] + \vec{a}_\theta [k_r (J_r k_\theta - J_\theta k_r)] - \vec{a}_\phi J_\phi . \qquad \text{(A-23)}$$

Equation (A-16), reduced to component form with the substitution of equations (A-22) and (A-23), is

$$J_r(a - ck_\theta^2) + J_\theta(ck_rk_\theta) + J_\phi(-bk_\theta) = \epsilon_o\omega_p^2 E_r \tag{A-24}$$

$$J_r(ck_\theta k_r) + J_\theta(a - ck_r^2) + J_\phi(bk_r) = \epsilon_o\omega_p^2 E_\theta \tag{A-25}$$

$$J_r(bk_\theta) + J_\theta(-bk_r) + J_\phi(a - c) = \epsilon_o\omega_p^2 E_\phi \tag{A-26}$$

The determinant of these three equations is

$$D = \begin{vmatrix} a - ck_\theta^2 & ck_rk_\theta & -bk_\theta \\ ck_\theta k_r & a - ck_r^2 & bk_r \\ bk_\theta & -bk_r & a - c \end{vmatrix} = a[(a-c)^2 + b^2]. \tag{A-27}$$

The three equations (A-24) through (A-26) are inverted to give the current distribution in terms of the three electric field components. In matrix form, the resulting equations are

$$\begin{bmatrix} J_r \\ J_\theta \\ J_\phi \end{bmatrix} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix} \begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix}$$

The resulting conductivity matrix is

$$\sigma = \frac{\epsilon_o\omega_p^2}{D} \begin{bmatrix} a(a-c) + [b^2 - c(a-c)]k_r^2 & [b^2 - c(a-c)]k_rk_\theta & abk_\theta \\ [b^2 - c(a-c)]k_rk_\theta & a(a-c) + [b^2 - c(a-c)]k_\theta^2 & -abk_r \\ -abk_\theta & abk_r & a(a-c) \end{bmatrix} \tag{Δ_3}$$

where a is defined by (A-17), b by (A-18), c by (A-19), k_r and k_θ by (A-20) and (A-21), and D by equation (A-27).

The conductivity matrix Δ_3 is very general, containing in its terms the effects of aprticle interaction. If the medium is assumed to be such that very few ion-electron, ion-neutral particle, or electron-neutral particle collisions occur, elements contained in the matrix are greatly simplified.

For no collisions,

$$f_{ie} = f_{in} = f_{en} = 0 .$$

Therefore

$$f_1 = f_2 = f_3 = 0 ,$$

and the three terms a, b, and c are found to be

$$a = i\omega$$

$$b = \frac{Mq}{m_e r^3} h$$

$$c = -\left(\frac{Mqh}{m_e r^3}\right)^2 \frac{1}{i\omega a_o} .$$

Then the various components of the matrix become

$$a(a-c) = -\omega^2 \left[1 - \frac{h^2}{\omega^2 a_o} \left(\frac{Mq}{m_e r^3}\right)^2\right]$$

$$b^2 - c(a - c) = \left(\frac{Mqh}{m_e r^3}\right)^2 \left\{1 + \frac{1}{a_o}\left[1 - \frac{h^2}{\omega^2 a_o}\left(\frac{Mq}{m_e r^3}\right)^2\right]\right\}$$

$$k_r k_\theta = \frac{2\lambda\sqrt{1-\lambda^2}}{h^2}$$

$$ab = i\omega h \frac{Mq}{m_e r^3}$$

$$D = i\omega\left\{-\omega^2\left[1 - \frac{h^2}{\omega^2 a_o}\left(\frac{Mq}{m_e r^3}\right)^2\right]^2 + \left(\frac{Mqh}{m_e r^3}\right)^2\right\}.$$

It is seen that a(a - c) and $b^2 - c(a - c)$ are entirely real while ab and D are purely imaginary. This causes $\sigma_{r\phi}$, $\sigma_{\theta\phi}$, $\sigma_{\phi r}$, and $\sigma_{\phi\theta}$ to be real, and σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta r}$, $\sigma_{\theta\theta}$, and $\sigma_{\phi\phi}$ to be imaginary. Also, it may be noted that $\sigma_{r\theta} = \sigma_{\theta r}$, $\sigma_{r\phi} = \sigma_{\phi r}$, $\sigma_{\theta\phi} = -\sigma_{\phi\theta}$.

$$b^2 - c(a - c) = \left(\frac{Mqh}{m_e r^3}\right)^2 \left\{1 + \frac{1}{a_o}\left[1 - \frac{h^2}{\omega^2 a_o}\left(\frac{Mq}{m_e r^3}\right)^2\right]\right\}$$

$$k_r k_\theta = \frac{2\lambda\sqrt{1-\lambda^2}}{h^2}$$

$$ab = i\omega h \frac{Mq}{m_e r^3}$$

$$D = i\omega\left\{-\omega^2\left[1 - \frac{h^2}{\omega^2 a_o}\left(\frac{Mq}{m_e r^3}\right)^2\right]^2 + \left(\frac{Mqh}{m_e r^3}\right)^2\right\}.$$

It is seen that $a(a - c)$ and $b^2 - c(a - c)$ are entirely real while ab and D are purely imaginary. This causes $\sigma_{r\phi}$, $\sigma_{\theta\phi}$, $\sigma_{\phi r}$, and $\sigma_{\phi\theta}$ to be real, and σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta r}$, $\sigma_{\theta\theta}$, and $\sigma_{\phi\phi}$ to be imaginary. Also, it may be noted that $\sigma_{r\theta} = \sigma_{\theta r}$, $\sigma_{r\phi} = \sigma_{\phi r}$, $\sigma_{\theta\phi} = -\sigma_{\phi\theta}$.

APPENDIX C

USEFUL PROPERTIES OF THE SPHERICAL BESSEL FUNCTIONS AND ASSOCIATED LEGENDRE POLYNOMIALS

SPHERICAL BESSEL FUNCTIONS

The spherical Bessel functions of the third kind are defined by

$$h_n^{(1)}(\rho) = \sqrt{\frac{\pi}{2\rho}}\, H^{(1)}_{n+\frac{1}{2}}(\rho)$$

$$h_n^{(2)}(\rho) = \sqrt{\frac{\pi}{2\rho}}\, H^{(2)}_{n+\frac{1}{2}}(\rho).$$

The Hankel functions $H^{(1)}_{n+\frac{1}{2}}(\rho)$ and $H^{(2)}_{n+\frac{1}{2}}(\rho)$ are linear combinations of the Bessel functions of the first and second kins, namely,

$$H^{(1)}_{n+\frac{1}{2}}(\rho) = J_{n+\frac{1}{2}}(\rho) + i\,N_{n+\frac{1}{2}}(\rho)$$

$$H^{(2)}_{n+\frac{1}{2}}(\rho) = J_{n+\frac{1}{2}}(\rho) - i\,N_{n+\frac{1}{2}}(\rho).$$

For large values of ρ, the approximations for the Spherical Hankel functions are, as $\rho \to \infty$,

$$h_n^{(1)}(\rho) \simeq \frac{1}{\rho}\,(-i)^{n+1}\, e^{i\rho}$$

$$h_n^{(2)}(\rho) \simeq \frac{1}{\rho}\,(i)^{n+1}\, e^{-i\rho}\;.$$

If $z_n(\rho)$ is either $h_n^{(1)}(\rho)$ or $h_n^{(2)}(\rho)$, then some useful recurrence

relations for the spherical functions are

$$\frac{2n+1}{\rho} Z_n(\rho) = Z_{n-1}(\rho) - Z_{n+1}(\rho)$$

$$\frac{d}{d\rho} [Z_n(\rho)] = Z_{n-1}(\rho) - \frac{n+1}{\rho} Z_n(\rho)$$

$$\frac{d}{d\rho} [Z_n(\rho)] = \frac{n}{\rho} Z_n(\rho) - Z_{n+1}(\rho)$$

$$\frac{d}{d\rho} [Z_n(\rho)] = \frac{n Z_{n-1}(\rho) - (n+1)Z_{n+1}(\rho)}{2n+1}$$

$$\frac{d^m}{d\rho^m} [\rho^{n+1} Z_n(\rho)] = \rho^{n+1} Z_{n-m}(\rho)$$

$$\frac{d^m}{d\rho^m} [\frac{1}{\rho^n} Z_n(\rho)] = (-1)^m \rho^{-n} Z_{n+m}(\rho),$$

for $n = 0, \pm 1, \pm 2, \ldots$, and $m = 1, 2, 3, \ldots$.

ASSOCIATED LEGENDRE POLYNOMIALS

The associated Legendre polynomial is defined by

$$P_n^m(\lambda) = (1-\lambda^2)^{m/2} \frac{d^m}{d\lambda^m} P_n(\lambda),$$

or

$$P_n^m(\lambda) = \frac{(1-\lambda^2)^{m/2}}{2^n n!} \frac{d^{n+m}(\lambda^2 - 1)^n}{d\lambda^{n+m}} .$$

Some recurrence relations which are useful in the calculation of the associated Legendre polynomials on a digital computer are, with the

notation of the functional dependence on λ omitted,

$$P_{n+1}^{m} = \frac{(2n+1)\lambda P_n^m - (n+m) P_{n-1}^m}{n-m+1}$$

$$P_{n+1}^{m} = \lambda P_n^m + (n+m)\sqrt{1-\lambda^2}\, P_n^{m-1}$$

$$\sqrt{1-\lambda^2}\, P_n^{m+1} = (n+m+1)\lambda P_n^m - (n-m+1) P_{n+1}^m$$

$$\sqrt{1-\lambda^2}\, P_n^{m+1} = 2m\lambda P_n^m - (n-m+1)(n+m)\sqrt{1-\lambda^2}\, P_n^{m-1}$$

$$\frac{m}{\sqrt{1-\lambda^2}} P_n^m = \frac{1}{2}\lambda \left[(n-m+1)(n+m)P_n^{m-1} + P_n^{m+1}\right] + m\sqrt{1-\lambda^2}\, P_n^m$$

$$(1-\lambda^2)\frac{dP_n^m}{d\lambda} = (n+1)\lambda P_n^m - (n-m+1) P_{n+1}^m$$

$$(1-\lambda^2)\frac{dP_n^m}{d\lambda} = (n+m) P_{n-1}^m - n\lambda P_n^m .$$

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13. ABSTRACT

A finite-difference approach to the description of the electromagnetic field structure within the region bounded by the earth and some outer spherical boundary is studied. Within this medium a current sheet and a free charge sheet are assumed to exist in a spherical geometry. The medium is assumed to be two-dimensionally inhomogeneous and to consist of a plasma gas. The earth is considered to be perfectly conducting, and the field distribution at the outer boundary is caused to assume the form of a system of spherical waves which are propagating outwardly.

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